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New universality of Lyapunov spectra in Hamiltonian systems

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Abstract. We show that new universality of Lyapunov spectra $\{\lambda_i\}$ exists in Hamiltonian systems with many degrees of freedom. The universality appears in systems which are neither nearly integrable nor fully chaotic, and it is different from the one which is obtained in fully chaotic systems on one-dimensional chains as follows. One is that the universality is found in a finite range of large i/N rather than the whole range, where N is the number of degrees of freedom. Another is that Lyapunov spectra are not straight, while fully chaotic systems give straight Lyapunov spectra even on the three-dimensional simple cubic lattice. The universality appears when quadratic terms of a potential function dominate higher terms, harmonic motions are hence regarded as the base of global motions.

1. Introduction

During the past few decades Hamiltonian systems with many degrees of freedom have been numerically investigated by integrating equations of motion. Hamiltonian systems being the foundation of statistical mechanics, one direction of the investigations is to check the ergodic property originating in Fermi–Pasta–Ulam problem [1–3]. Another direction is to study their dynamical properties, which are transition from nearly integrable systems to stronger chaotic ones [4, 5], dynamical properties of phase transition [6, 7], structure of phase spaces [5, 8], etc.

Here we focus on dynamical properties, in particular, universal structures of phase spaces which are not affected by details of systems. One of the structures is a self-similar structure which is based on the Poincaré–Birkhoff theorem [9] in nearly integrable systems. Although this theorem is available in systems with two degrees of freedom, the self-similar structure is also supposed in systems with many degrees of freedom.

Lyapunov spectrum is usually used to study instability along a sample orbit. Moreover, it is useful to study the structure of phase spaces both in dissipative [10, 11] and Hamiltonian [5, 8] systems with many degrees of freedom, since it includes information on all directions in phase space. Although Lyapunov spectra in a system reveal dynamical properties of the system, we are interested further in properties which are not affected by details of systems. To detect such properties a useful approach is to find the universal form of Lyapunov spectra which is obtained in all the systems and whose cause indicates the properties. A universal form of Lyapunov spectra $L(i/N) = \lambda_i$ is reported in Hamiltonian systems which are one-dimensional chains consisting of nonlinear oscillators [12], where N is the number of degrees of freedom. This universality gives the straight form for Lyapunov spectra, namely

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$L(i/N) = a + b \cdot i/N$, and the straight form is also obtained with random matrices [12, 13]. Consequently, systems having straight Lyapunov spectra are regarded as fully chaotic ones which have no structure in phase spaces.

In this paper we show new universality of Lyapunov spectra appearing in systems with moderate strength of chaos which are neither fully chaotic nor nearly integrable, and study the cause of the universality. Here we define the word 'universality of Lyapunov spectra' as $L(i/N)$ approximately takes the same form in a finite range of large i/N regardless of total energy and details of systems. Systems with moderate strengths of chaos are interesting for the following two reasons. One is that they have some unsolved problems and show interesting phenomena, for instance, second-order phase transition [14], and the other is that the structure of phase spaces has not been understood while it is understood in systems which are nearly integrable or fully chaotic. Models investigated in this paper consist of nonlinear oscillators with nearest-neighbour interactions, and each oscillator is on a lattice point of the three-dimensional simple cubic lattice.

This paper is constructed as follows. We introduce five models in section 2. They are used to confirm the new universality of Lyapunov spectra which appears in a wide class of Hamiltonian systems. We show Lyapunov spectra which is yielded by using random matrices with temporal δ- or exponential correlations in section 3. The Lyapunov spectra are straight even in systems which are on the three-dimensional lattices when the systems are fully chaotic. In sections 4 and 5, we show that Lyapunov spectra for the five systems are not straight and have universality, and that the systems are neither fully chaotic nor nearly integrable. In section 5 we show that the new universality appears when quadratic terms of a potential function, U_2, dominate higher terms, U_4, namely U_2/U_4 takes large values. Section 6 is devoted to a summary and discussions.

2. Models

We introduce five model Hamiltonians each of which represents a system being on the three-dimensional simple cubic lattice with nearest-neighbour interactions and periodic boundary condition. All Hamiltonians consist of kinetic and potential terms,

$$H(q, p) = K(p) + U(q) \tag{1}$$

where the kinetic term is

$$K(p) = \sum_{j=1}^{N} \tfrac{1}{2} p_j^2 \tag{2}$$

and N is the number of degrees of freedom, namely $N = L^3$ where L is the linear size of the lattice.

One of the models is called the XY model and is expressed as follows

$$U_{XY}(q) = \sum_{\langle ij \rangle} [1 - \cos(q_i - q_j)] \qquad q_j \in [0, 2\pi) \tag{3}$$

where the summation $\sum_{\langle ij \rangle}$ takes over all the pairs of nearest-neighbour lattice points i and j.

The following three systems are expressed as

$$U(q) = \sum_{\langle ij \rangle} \tfrac{1}{2}(q_i - q_j)^2 + \sum_{j=1}^{N} V(q_j) \tag{4}$$

and they are distinguished by their potential functions $V(q)$. In the double well (DW) model

$$V_{\mathrm{DW}}(q) = -\tfrac{1}{2}q^2 + \tfrac{1}{4}q^4 \tag{5}$$

in the single well (SW) model

$$V_{\mathrm{SW}}(q) = \tfrac{1}{2}q^2 + \tfrac{1}{4}q^4 \tag{6}$$

and in the Lorentzian (LO) model

$$V_{\mathrm{LO}}(q) = \frac{q^2}{1+q^2}. \tag{7}$$

The last one has interactions of FPU-β type (3DFPU)

$$U_{\mathrm{3DFPU}} = \sum_{\langle ij\rangle} \left[\frac{k}{2}(q_i - q_j)^2 + \frac{1}{4}(q_i - q_j)^4 \right]. \tag{8}$$

This model is used in section 5 to determine which term is dominant when the new universality appears.

Numerical integrations of Hamiltonian equations of motion,

$$\frac{\mathrm{d}q_j}{\mathrm{d}t} = \frac{\partial H(q,p)}{\partial p_j} \qquad \frac{\mathrm{d}p_j}{\mathrm{d}t} = -\frac{\partial H(q,p)}{\partial q_j} \qquad (j = 1, 2, \ldots, N) \tag{9}$$

are performed with fourth-order symplectic integrator with the fixed time slice $\Delta t = 0.01$. Accuracy of total energy is $\Delta E/E \sim \mathrm{O}((\Delta t)^4)$ where ΔE and E are error and an initial value of total energy, respectively.

3. Lyapunov spectra with random matrices

As mentioned in the introduction, Lyapunov spectra $L(i/N)$ calculated with random matrices are straight in one-dimensional chains. The Lyapunov spectrum is a set of Lyapunov exponents $\{\lambda_i\}$ $(i = 1, 2, \ldots, D)$, where D is the dimension of phase space and the exponents are put in order as $\lambda_i \geqslant \lambda_{i+1}$. The summation of λ_i up to n, $\sum_{i=1}^{n} \lambda_i$, indicates linear instability of the n-dimensional volume element V_n in phase space along a sample orbit. Namely, V_n diverges or converges as

$$V_n(t) \sim \exp[(\lambda_1 + \lambda_2 + \cdots + \lambda_n)t] \tag{10}$$

where t represents time. Each of Hamiltonian systems with N degrees of freedom has $2N$ Lyapunov exponents which satisfy the following relations induced by symplectic properties

$$\lambda_{2N-i+1} = -\lambda_i \qquad (i = 1, 2, \ldots, N). \tag{11}$$

Hence, we have only to observe the first half of the Lyapunov spectrum when we consider Hamiltonian systems. Details of Lyapunov spectrum are reviewed in [15] and references therein.

The purpose of this section is to show that fully chaotic systems have straight Lyapunov spectra even in the three-dimensional simple cubic lattice. We describe how we calculate Lyapunov spectra with random matrices, and then the Lyapunov spectra are shown.

Lyapunov spectra indicate linear instability of a sample orbit, and they are calculated from linearized equations of motion,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \delta q \\ \delta p \end{pmatrix} = \begin{pmatrix} 0 & 1_N \\ -A(q) & 0 \end{pmatrix} \begin{pmatrix} \delta q \\ \delta p \end{pmatrix} \tag{12}$$

where $(\delta q, \delta p)$ is a tangent vector, 1_N is the unit matrix of $N \times N$, and (i, j) element of the matrix $A(q)$ is

$$A_{ij}(q) = \frac{\partial^2 U(q)}{\partial q_i \partial q_j}. \tag{13}$$

Here we used the form of our Hamiltonians, equations (1) and (2). Temporal evolution of the matrix $A(q)$ is determined by temporal evolution of $q_j(t)$'s, and in this section we assume that $q_j(t)$'s are independent random variables with δ- or exponential correlations, namely

$$C_{ij}(t) \propto \delta_{ij}\delta(t) \qquad \text{or} \qquad \delta_{ij}\mathrm{e}^{-\alpha t}. \tag{14}$$

$C_{ij}(t)$ is the correlation function between $q_i(t)$ and $q_j(t)$, and it is defined as

$$C_{ij}(t) = \overline{\Delta q_i(t)\Delta q_j(0)} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{d}t'\, \Delta q_i(t+t')\Delta q_j(t') \tag{15}$$

and

$$\Delta q_j(t) = q_j(t) - \lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{d}t\, q_j(t). \tag{16}$$

We show Lyapunov spectra in figure 1 which are calculated with random matrices having the δ-correlation, and the Lyapunov spectra are straight as they are in one-dimensional chains. We take random variables from the uniform distribution, and ranges of $q_j(t)$'s are $[-\pi, \pi)$ for the XY model, and $[-3, 3]$ for DW and SW models.

Next we change δ-correlation into exponential, and we set $\alpha = 0.4, 0.6, 0.8$ and 1.0 where α is the reciprocal number of correlation time of $q_j(t)$ (see equation (14)). Figure 2 shows the Lyapunov spectra obtained by using random matrices with exponential correlation, and the Lyapunov spectra are also straight in the region of $0.2 \lesssim i/N \leqslant 1$. Consequently, we suppose that finite correlation time does not affect the straightness of Lyapunov spectra.

Random matrices with δ- or exponential correlation yield straight Lyapunov spectra, and fully chaotic systems hence give the straightness even in the three-dimensional simple cubic lattice. This result is used later to distinguish that our systems evolved by Hamiltonian equations of motion from fully chaotic ones.

4. New universality of Lyapunov spectra

In this section, we show that new universality of Lyapunov spectra exists in systems which are neither fully chaotic nor nearly integrable and which are in the thermodynamic limit ($N \to \infty$) through giving the following four results. (i) The degrees of freedom $N = 4^3$ is high enough to reach the thermodynamic limit for Lyapunov spectra. (ii) Forms of Lyapunov spectra are invariant with respect to energy in each of the four models which are XY, DW, SW and LO in a finite range of large i/N. (iii) The invariant forms of the four systems are in good agreement, and they are not approximated by the straight line. Here we conclude that new universality of Lyapunov spectra is found. (iv) Appearance of the new universality is not limited in nearly integrable systems.

Lyapunov spectra for the XY model are shown in figure 3(a) in which points and dots represent that the degrees of freedom are $N = 4^3$ and 10^3 respectively. Numbers in the figure are values of energy density E/N. The Lyapunov spectra do not correspond with each other even though they have the same energy density.

Let us note that the form of the Lyapunov spectrum $L(i/N)$ is determined by ratios between Lyapunov exponents λ_i rather than their absolute values. Accordingly, we may

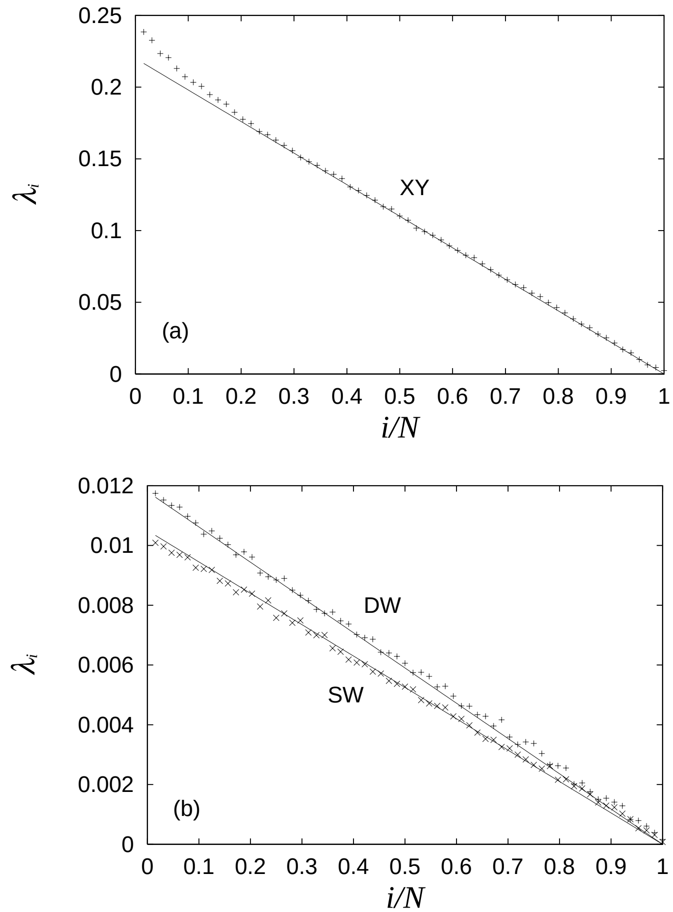


Figure 1. Lyapunov spectra yielded by using random matrices with δ-correlation for XY, DW and SW models. (a) XY model. (b) DW ($\Diamond$) and SW ($+$) models. They are straight even in systems on three-dimensional lattices as they are in one-dimensional chains. Random variables follow uniform distributions. The full lines are guides for eyes.

uniformly scale the Lyapunov spectrum up or down from $L(i/N)$ to $\gamma L(i/N)$, where γ is arbitrarily picked for each spectrum. To multiple $L(i/N)$ by γ corresponds to changing the timescale from t into t/γ. Scaled Lyapunov spectra are shown in figure 3(b), and their forms holds regardless of the degrees of freedom, where scale factors are $\gamma = \frac{1}{1.1}$ and 1.0 for each spectrum in $N = 4^3$ and 10^3 respectively. Since the thermodynamic limit of Lyapunov spectra [16, 17] exists, we suppose that the system reaches the thermodynamic limit even $N = 4^3$ concerning Lyapunov spectra. We therefore set $N = 4^3$, and use scaled Lyapunov spectra without comment hereafter.

Invariance of $L(i/N)$ with respect to energy is shown in figure 4 for the four models. In each of the models, Lyapunov spectra are in good agreement among various values of energy in a range of large i/N in the middle energy regime. Scale factor γ and the ranges of i/N where the invariance appears are arranged in tables A1 and A2, respectively. We remark that the invariance breaks or is strictly limited in a narrow range of i/N when energy is too high or low, that are $E/N = 100$ in XY and DW, and $E/N = 1.0$ in SW.

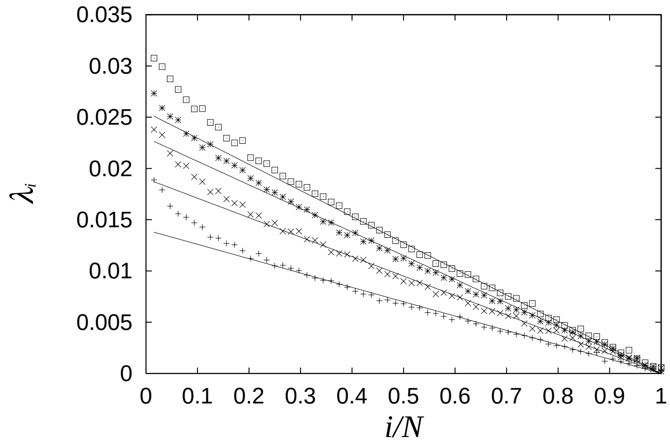


Figure 2. Lyapunov spectra yielded by using random matrices with exponential correlation for DW. α, the reciprocal number of correlation time, is 0.4, 0.6, 0.8 and 1.0 from lower to upper. Spectra show straight behaviour as figure 1 in the region of $0.2 \lesssim i/N \leqslant 1$. The full lines are guides for eyes.

Next we show the invariant forms of Lyapunov spectra of the four models together in figure 5. Scale factor γ is arranged in table A3. The four spectra in figure 5 coincide in a range of large i/N ($0.4 \lesssim i/N \leqslant 1$), while the full line, which is obtained with random matrices, approximates them in a narrower range ($0.7 \lesssim i/N \leqslant 1$). That is, the invariant forms do not depend on details of models and are not straight, and consequently, Lyapunov spectra of the four models have universality which is different from the one obtained in fully chaotic systems.

Moreover, we show that the new universality appears even when systems are not nearly integrable. We confirm that systems are not nearly integrable through showing that KAM tori, many of which survive in nearly integrable systems [18], are not observed effectively. We use the fact that linear instability is suppressed around KAM tori since an orbit behaves like a regular one, while enhanced in chaotic sea. Accordingly, in nearly integrable systems, intermittency of local Lyapunov exponent $\lambda_1^{\mathrm{loc}}(n)$ occurs which is defined as follows

$$\lambda_1^{\mathrm{loc}}(n) = \frac{1}{\tau}\int_{n\tau}^{(n+1)\tau} \lambda_1(t)\,\mathrm{d}t \tag{17}$$

$$\lambda_1(t) = \frac{\mathrm{d}}{\mathrm{d}t}\log|X(t)| \tag{18}$$

where $X(t) = (\delta q, \delta p)$ is a $2N$-dimensional tangent vector following linearized Hamiltonian equations of motion, equation (12). Figure 6 shows two time series of local Lyapunov exponent for $E/N = 1.0$ and 3.0 in the XY model, both of which yield the new universal Lyapunov spectra. Here $N = 10^3$. Intermittency is found for $E/N = 1.0$ but not for $E/N = 3.0$, and hence the existence of KAM tori does not seem to be related to the new universality. Consequently, the universality appears in systems with moderate strength of chaos which are between nearly integrable and fully chaotic.

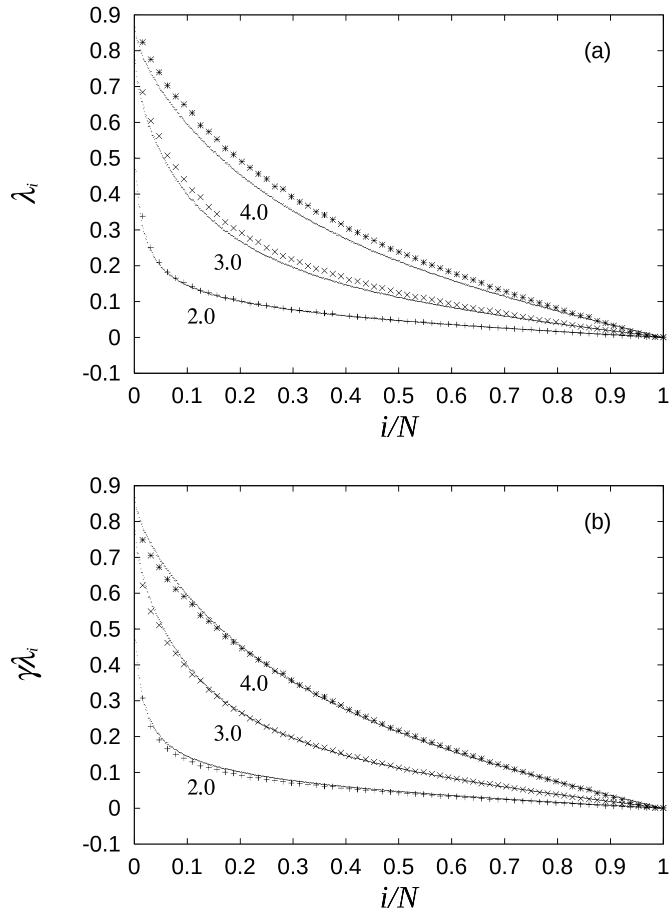


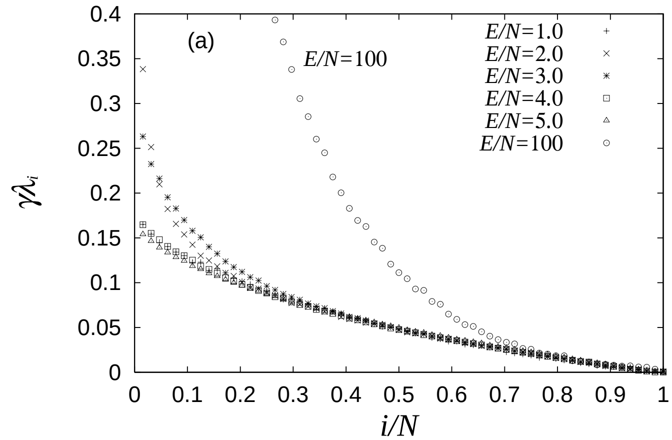
Figure 3. Dependence on degrees of freedom N. Points and dots are Lyapunov spectra for $N = 4^3$ and 10^3 respectively. Numbers in the figure represent values of energy density E/N. (a) Non-scaled. (b) Vertical axis is scaled for $N = 4^3$ and the scale factor $\gamma = \frac{1}{1.1}$ for all values of energy. In each energy, scaled Lyapunov spectrum for $N = 4^3$ is in good agreement with one for $N = 10^3$.

Table 1. Behaviour of Lyapunov spectra in 3DFPU model. k and E/N are coupling constant and energy density, respectively. The sign S means straight behaviour of the Lyapunov spectrum, while the sign C curved behaviour. The forms of curved Lyapunov spectra are in good agreement with the universal form.

$k \setminus E/N$	1.0	2.0	3.0
0.4	S	S	S
0.7	C	S	S
1.0	C	C	S

5. 3DFPU model and quadratic interactions

To probe what conditions produce the universality, we study the 3DFPU model with various values of energy density E/N and coupling constant k (see equation (8)). Lyapunov spectra



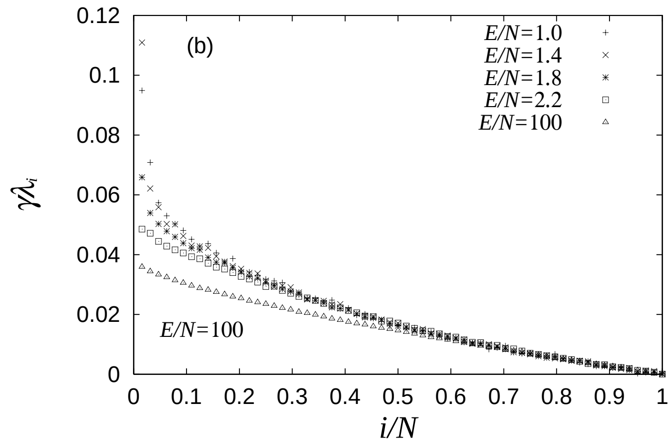


Figure 4. Lyapunov spectra for various values of energy in four Hamiltonian systems. (a) XY model. (b) DW model. (c) SW model. (d) LO model. Forms of Lyapunov spectra are the same in a range of large i/N in the middle energy regime in each model. Scale factor γ and the range of i/N giving the invariant form are arranged in tables A1 and A2.

Table 2. Ratios between time averages of the quadratic term U_2 and the quartic terms U_4 of potential function in 3DFPU model. We subtract 1.50 from each of the ratios to make a threshold clear, namely values arranged in this tables are $\overline{U}_2/\overline{U}_4 - 1.5$. The positive values are found at the places where the sign C appears in table 1.

$k \setminus E/N$	1.0	2.0	3.0
0.4	−0.60	−0.91	−1.02
0.7	0.33	−0.33	−0.62
1.0	1.55	0.37	−0.09

for the 3DFPU model are shown in figure 7 with a Lyapunov spectrum for the XY model belonging in the universality, and they are classified into group C and group S. Group C includes the universal Lyapunov spectrum which is in good agreement with curved three spectra for 3DFPU, and other spectra for 3DFPU are straight and belong to group S. The

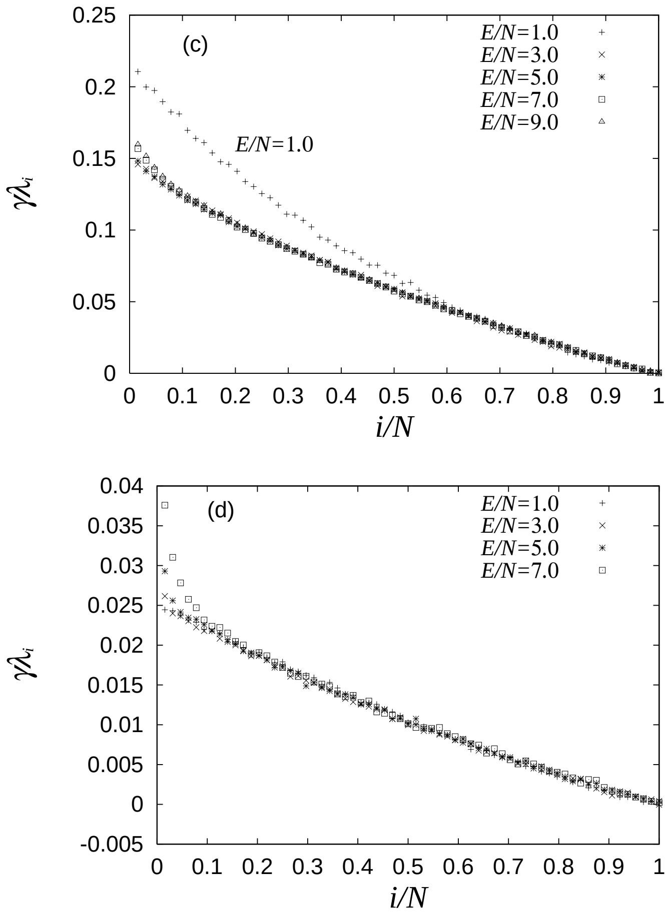


Figure 4. (Continued)

classification is arranged in table 1 with respect to k and E/N. Lyapunov spectra show curved forms when k is large and E/N is low, we hence conjecture that quadratic terms of potential function are dominant rather than quartic terms when the universality appears.

This conjecture is verified by taking ratios between time averages of U_2 and U_4, namely $\overline{U_2}/\overline{U_4}$, where

$$U_2 = \sum_{\langle ij \rangle} \frac{k}{2}(q_i - q_j)^2 \tag{19}$$

$$U_4 = \sum_{\langle ij \rangle} \tfrac{1}{4}(q_i - q_j)^4 \tag{20}$$

and

$$\overline{U}_n = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{d}t\, U_n(t) \qquad (n = 2, 4). \tag{21}$$

The ratios are arranged in table 2, and they are large at the places where the sign C appears in table 1. Consequently, the conjecture is verified.

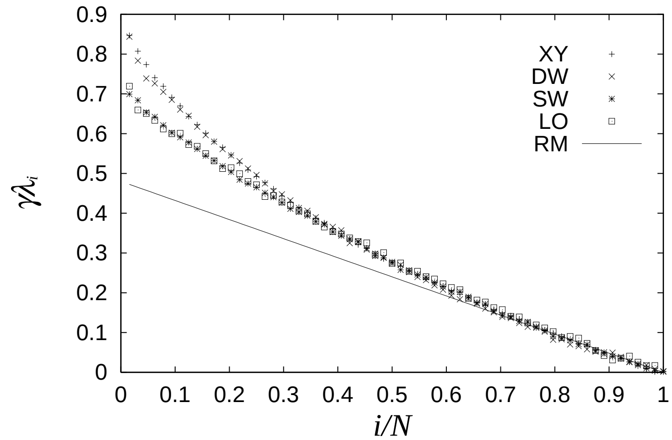


Figure 5. Universal behaviour of scaled Lyapunov spectra of four models. The Lyapunov spectra coincide well among the four models in a range of large i/N, namely $0.4 \lesssim i/N \leqslant 1$, and they are not approximated by the straight line, which approximates them only in the range of $0.7 \lesssim i/N \leqslant 1$. The straight line is obtained with random matrices. Scale factor γ is arranged in table A3.

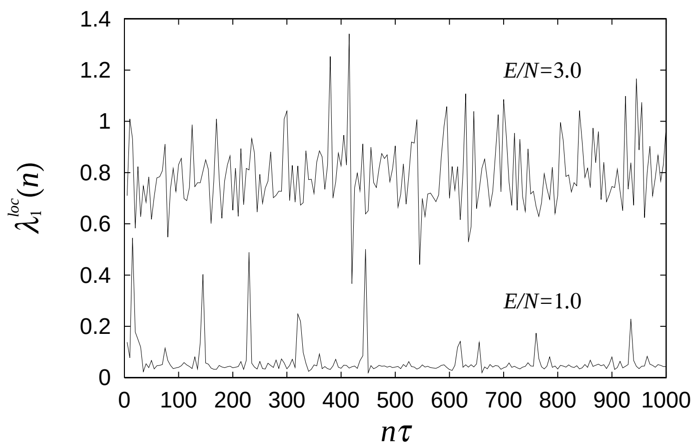


Figure 6. Temporal evolutions of local Lyapunov exponent $\lambda_1^{\mathrm{loc}}(n)$ in the XY model. $N = 10^3$. We set $\tau = 10$. Lower and upper curves take $E/N = 1.0$ and 3.0 respectively, both of which yield the new universal Lyapunov spectra. Intermittency is found for $E/N = 1.0$ but not for $E/N = 3.0$.

6. Summary and discussions

In order to consider the structure of phase spaces in Hamiltonian systems with moderate strength of chaos, we numerically investigated Lyapunov spectra $\{\lambda_i\}$ for five Hamiltonian systems with many degrees of freedom. We showed the existence of universality of Lyapunov spectra, which is defined as Lyapunov spectra approximately take the same form regardless of energy and details of the systems. The universality gives a curved

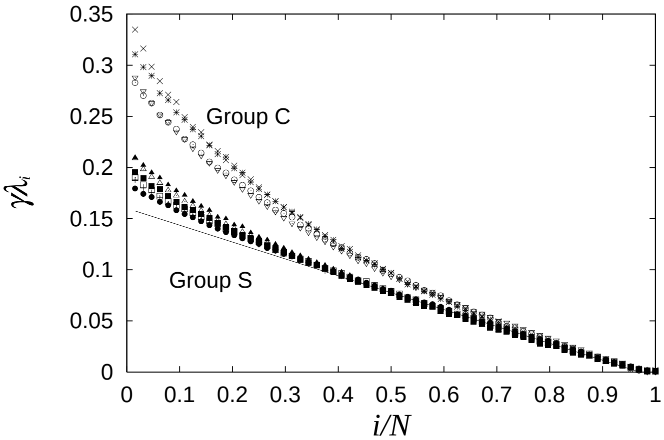


Figure 7. Lyapunov spectra in 3DFPU model. They are classified into group C and group S. Group C includes the universal spectrum which is in agreement with three spectra for 3DFPU, while group S consists of six spectra for 3DFPU which are straight. The straight line is a guide for eyes. The classification is arranged in table 1 with respect to k and E/N. Scale factor γ shown in table A4.

form for Lyapunov spectra, and hence it is different from the one obtained in fully chaotic systems.

A feature of the new universality is that it appears in a finite range of large i/N, where N is degrees of freedom, and accordingly, properties depending on energy or details of models affect forms of Lyapunov spectra only in the range of small i/N where is out of the universality. In other words, we only have to focus on the range of small i/N when we are interested in such individual properties.

We studied what conditions induce the universality to understand the cause of it. We showed that quadratic terms of a potential function dominate higher terms when the universality appears, harmonic motions are therefore regarded as the base of global motions in phase spaces.

We geometrically interpret the harmonic motions and that the universality appears in a finite range of i/N as follows. Harmonic motions occur in high-dimensional subspaces of phase spaces, which correspond to the finite range of i/N, and structure of phase space consists of chaotic sea and wrecks of n-dimensional tori, where $n \leqslant N$ and n may change for each torus. Here n-dimensional torus means, roughly speaking, direct product of T^n and $(N-n)$-dimensional instability, which is hyperbolic or complex. Note $n = N$ is KAM torus.

We need further analyses to confirm whether this interpretation is valid or not, and to understand the origin of the new universality. We give two approaches which are geometrical and analytical.

A geometrical approach uses resonance of instability along an orbit. Negative curvature of potential function induces positive Lyapunov exponent. On the other hand, even the curvature is always positive, resonance of instability along a sample orbit gets the largest Lyapunov exponent to be positive [19]. Here let us assume that this resonance theory can be extended not only to the largest Lyapunov exponent but to all the exponents. If the resonances are yielded by harmonic motions in high-dimensional subspace of phase space, then universality of Lyapunov spectra may be obtained because harmonic motions are independent of details of models.

An analytical approach is to calculate the decay rate spectrum of harmonic motions. In a dissipative system, behaviour of Lyapunov spectrum agrees with the decay rate spectrum of the linear fluctuation modes from the stationary solution [10]. This suggests that to analyse decay rate spectrum it is useful to understand the behaviour of Lyapunov spectrum. Hamiltonian systems are regarded as dissipative systems when we observe only subspaces of phase spaces, and hence the decay rate spectrum must be useful because the new universality appears only in high-dimensional subspaces of phase spaces.

Our aim is to understand the global structure of phase spaces in Hamiltonian systems with moderate strength of chaos. We must be allowed to come near the goal by understanding the cause of the new universality, because ratios between Lyapunov exponents, which determine the form of the Lyapunov spectrum, seem to concern global structure of phase spaces. The new universality of Lyapunov spectra is an important clue to reveal the global structure.

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Appendix. Tables of scale factor γ

Table A1. Scale factor γ in figure 4.

XY		DW		SW		LO	
E/N	γ	E/N	γ	E/N	γ	E/N	γ
1.0	3.4	1.0	6.0	1.0	12	1.0	1.0
2.0	1.0	1.4	2.4	3.0	2.65	3.0	1.2
3.0	1/2.6	1.8	1.4	5.0	1.6	5.0	1.4
4.0	1/5.0	2.2	1.0	7.0	1.2	7.0	1.95
5.0	1/5.5	100	1/11	9.0	1.0		
100	4.0						

Table A2. Rough estimations of ranges of i/N where universal behaviour of Lyapunov spectra appears in four models.

Model	*XY*	DW	SW	LO
i/N	[0.4, 1]	[0.31, 1]	[0.16, 1]	[0.11, 1]

Table A3. Scale factor γ in figure 5.

Model	XY	DW	SW	LO
E/N	6.0	2.0	3.0	3.0
γ	1.0	18	12.7	33

Table A4. Scale factor γ in figure 7.

$k \setminus E/N$	1.0	2.0	3.0
0.4	1.0	0.75	0.66
0.7	1.7	0.91	0.74
1.0	2.25	1.4	0.92

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